

# STEADY-STATE MARGINALIZED PARTICLE FILTER FOR ATTITUDE ESTIMATION

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## ABSTRACT

A marginalized particle filter (MPF) is designed for attitude estimation problem. Unit quaternions are used to parameterize rotations. The linear structure in the gyroscope bias dynamics enables us to completely decouple its evolution from quaternion particles. We further show that the linear part of the proposed MPF reaches a steady state, similar to what Kalman filter does for controllable and observable linear stochastic systems. Although the steady-state MPF is similar to the particle filter in structure, it has two advantages: (i) the theoretical superiority of marginalizing linear substructure, and (ii) the reduction in total computational time. Numerical simulations are performed to demonstrate the performance of the proposed filter.

## 1 INTRODUCTION

Attitude estimation can be traced back to 1965 when Wahba posed the question to estimate the attitude of a spacecraft in the sense of least squares given noisy observations [1], resolved in both of the body frame and the reference frame. Some well-known estimators that robustly solve Wahba's problem are Davenport's  $q$  method [2], and the QUaternion ESTimator (QUEST) [3].

For attitude estimation with increased accuracy, strap-down gyroscopes are used in combination with vector measurements. Because of the nonlinear nature of the attitude kinematic equation and measurement model, the problem is a nonlinear state estimation problem, which is typically tackled by Bayesian inference. The extended Kalman filter (EKF) was first studied in attitude estimation and in particular, Lefferts *et. al.* proposed a multiplicative error approach in which the error quaternion and the gyroscope bias are defined as the filter states [4, 5]. The

drawback of EKF is that the mean and covariance of the state is propagated analytically through the first-order linearization of the nonlinear dynamics, which may introduce large approximation errors and lead to sub-optimal filter performance. The unscented Kalman filter (UKF) is superior to EKF in terms of capturing the posterior mean and covariance of the state distribution (accurately up to the 3rd order) [6]. The UKF formulation for attitude estimation has been proposed by Crassidis *et. al.* [7].

Due to recent development in computational power, the use of particle filters (PF) gained much traction and became practical for a broad area of applications. PF computes the posterior state distribution by drawing random samples of the state vector (termed as particles) and evaluates the likelihood of getting the actual system measurements conditioned on each particle [8]. Cheng *et. al.* applied a bootstrap particle filter for sequential spacecraft attitude estimation [9]. Because of the high dimensionality of the state vector, a prohibitively large number of particles are needed to span the state space to support the state distribution. In contrast to this approach, Oshman *et. al.* reduces the computational burden by sampling only the attitude of the spacecraft and using a genetic algorithm to estimate the gyro bias [10].

If there exists a linear sub-structure inherent in the nonlinear dynamics, it is possible to marginalize out the linear state variables and estimate them instead with the Kalman filter (KF) while the nonlinear state variables are estimated using the PF. This powerful combination of PF and KF, called the marginalized particle filter (MPF) or the Rao-Blackwellized particle filter, can effectively increase the estimation accuracy and possibly reduce the computations [11]. The scheme has been directly applied for attitude estimation by Liu *et. al.* [12].

In this work, we further exploit the underlying linear-substructure and show that the linear state evolution is completely independent of the nonlinear part. The organization of the paper is as follows. In Sec.2, recursive Bayesian filtering and sequential-importance-resampling particle filter are reviewed. In Sec.2.2, the general marginalized particle filter is reviewed. In Sec.3, the quaternion kinematics along with the gyroscope and vector measurement models are presented. Then the marginalized particle filter formulation for attitude estimation is proposed. In Sec.4 an algorithmic comparison is performed for the proposed filter with the PF. In Sec.5, a numerical study is performed to show superior performance of MPF over PF and EKF.

## 2 BAYESIAN FILTERING

In this section, the sequential-importance-resampling (SIR) particle filter and the marginalized particle filter are reviewed. A general discrete-time nonlinear model is given by

$$\begin{aligned} x_{t+1} &= F_t(x_t, u_t, w_t) \\ y_t &= H_t(x_t, e_t) \end{aligned} \quad (1)$$

where  $x_t$  represents the system state at time  $t$ ,  $u_t$  and  $y_t$  are the input and output, respectively,  $w_t, e_t$  are the zero-mean process and measurement noises with mutually independent distributions. The Markov assumption is made that past and future data are independent if one knows the current state  $x_t$ . The corresponding Baye's net representation is shown in Figure 1. The Bayesian filtering problem is to compute the a-posteriori distribution  $P(x_t|Y_t)$ , where  $Y_t = \{y_k\}_{k=0}^t$  is the set of measurements up to and the current time,  $t$

Table 1 depicts the basic Bayes filter. It possesses two essential steps:

*Update:* compute the a-posteriori distribution of the filter state by incorporating the most recent measurement

*Propagation:* predict the a-priori distribution of the filter state using the process model and the current state distribution

The basic Bayes filter can only be implemented under the following two circumstances, either the state of interest is restricted in a finite state space, or the system process and measurement models are linear and the noises are zero-mean Gaussian distributed. The former can be done by the discrete Bayes filter in which the propagation requires a finite sum instead of the integration. The latter is dealt with by the well-known Kalman filter algorithm which carries out the integral in closed form.

<b>Initialization</b>	initialize $P(x_0)$ ,
<b>Measurement</b>	measure $y_t$ ,
<b>Update</b>	$P(x_t Y_t) = \eta P(y_t x_t)P(x_t Y_{t-1})$ ,
<b>Propagation</b>	$P(x_{t+1} Y_t) = \int P(x_{t+1} x_t, u_t)P(x_t Y_t)dx_t$ ,

Table 1. The basic Bayes filter algorithm.  $\eta$  is a normalization factor.

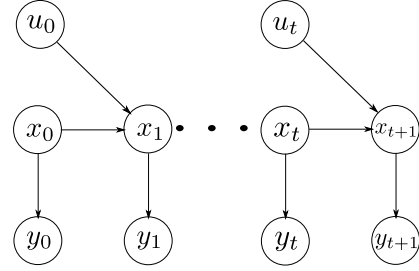


Figure 1. The Bayes' net representation of the underlying Markov assumption

### 2.1 Particle filter

In contrast, particle filters can deal with circumstances other than the two aforementioned. The idea, based on Monte Carlo methods, uses a large number of particles  $\{x_t^{(i)}\}_{i=1}^N$  to approximate the state distribution  $P(x_t)$ . In our work,  $x_t^{(i)}$  represents the  $i$ -th particle, out of  $N$ , at time  $t$ . It is also related to sequential importance sampling, the simplest form of which requires the particles be drawn from the a-priori distribution  $P(x_t|Y_{t-1})$ , denoted as  $\{x_{t|t-1}^{(i)}\}_{i=1}^N$  i.e. the set of  $N$  particles at time  $t$ , given measurements made up to the  $t-1$  time step. After obtaining the most recent measurement  $y_t$ , we can evaluate the importance weights,

$$\begin{aligned} w_t^{(i)} &\propto P(y_t|x_{t|t-1}^{(i)}) \\ w_t^{(i)} &\leftarrow w_t^{(i)} / \sum_{i=1}^N w_t^{(i)} \end{aligned} \quad (2)$$

That is, the importance weight is proportional to (denoted by  $\propto$ ) the likelihood of getting the measurement for a given instantiation of the state i.e. a particle, and the sum of weights is normalized to 1. The Monte Carlo estimate of the a-posteriori distribution is then

$$P(x_t|Y_t) = \sum_{i=1}^N w_t^{(i)} \delta(x_t - x_{t|t-1}^{(i)}) \quad (3)$$

where  $\delta(\bullet)$  denotes the Dirac delta. Usually, a resampling step is performed to refocus the particle set to regions in state space with

<b>Initialization</b>	initialize the particles, $x_{0 t-1}^{(i)} \sim P(x_0), \quad \forall i = 1, \dots, N$
<b>Measurement</b>	measure $y_t$
<b>Update</b>	assign weights $w_t^{(i)}$ to each particle, $w_t^{(i)} \propto P(y_t   x_{t t-1}^{(i)}) w_{t-1}^{(i)}$ and $\sum_{i=1}^N w_t^{(i)} = 1$
<b>A-posteriori estimate</b>	$\hat{x}_{t t} = \sum_{i=1}^N w_t^{(i)} x_{t t-1}^{(i)}$
<b>Resampling</b>	if $1 / \sum_{i=1}^N w_t^{(i)2} < N_{threshold}$ , resample $N$ particles so that, $Pr(x_{t t}^{(i)} = x_{t t-1}^{(j)}) = w_t^{(j)}$ reweight $w_t^{(i)} = 1/N$
<b>Propagation</b>	sample $x_{t+1 t}^{(i)}$ , $x_{t+1 t}^{(i)} \sim P(x_{t+1}   x_{t t}^{(i)}, u_t)$

Table 2. The sequential importance resampling particle filter algorithm. The symbol  $\sim$  denotes drawing a random sample from the target distribution.

high a-posteriori probability. A detailed derivation of particle filter can be found in [13]. An algorithmic form of the sequential importance resampling particle filter is summarized in Table 2.

## 2.2 Marginalized particle filter

The main idea behind the marginalized particle filter is to take advantage of the linear sub-structure in the filter dynamics, and marginalize out the corresponding linear state variables and estimate them using the Kalman filter [14]. If we partition the state variables as,

$$x_t = \begin{bmatrix} x_t^n \\ x_t^l \end{bmatrix} \quad (4)$$

where  $x_t^n$  and  $x_t^l$  are the nonlinear and linear parts of the state. The process and measurement models are given by,

$$\begin{aligned} x_{t+1}^n &= f_t^n(x_t^n) + A_t^n(x_t^n)x_t^l + G_t^n(x_t^n)w_t^n \\ x_{t+1}^l &= f_t^l(x_t^n) + A_t^l(x_t^n)x_t^l + G_t^l(x_t^n)w_t^l \\ y_t &= h_t(x_t^n) + C_t(x_t^n)x_t^l + e_t \end{aligned} \quad (5)$$

Compared with the general model (Eqn. (1)), here we assume that the linear state variables and the noises appear affinely in the filter model.

If we rewrite the a-posteriori distribution using the chain rule, we get

$$P(x_t | Y_t) = P(x_t^n, x_t^l | Y_t) = P(x_t^l | x_t^n, Y_t) P(x_t^n | Y_t) \quad (6)$$

We still use the SIR particle filter to approximate  $P(x_t^n | Y_t)$ . However, if  $P(x_0^l)$  is Gaussian distributed,  $P(x_t^l | x_t^n, Y_t)$  can be computed recursively by the Kalman filter. For each pair of instantiations  $\{x_t^{n,(i)}, x_{t+1}^{n,(i)}\}$ , Eqn. (5) becomes a linear system with two measurements  $z_t^1$  and  $z_t^2$ ,

$$\begin{aligned} x_{t+1}^l &= f_t^l + A_t^l x_t^l + G_t^l w_t^l \\ \underbrace{y_t - h_t}_{z_t^1} &= C_t x_t^l + e_t \\ \underbrace{x_{t+1}^n - f_t^n}_{z_t^2} &= A_t^n x_t^l + G_t^n w_t^n \end{aligned} \quad (7)$$

Therefore, in the marginalized particle filter, we keep track of a set of  $N$  particles representing the nonlinear partition of the state  $\{x_t^{n,(i)}\}$ , and  $N$  Kalman filters  $\{x_t^{l,(i)}, P_t^{(i)}\}$ , where  $x_t^{l,(i)}, P_t^{(i)}$  are the mean and the covariance of the  $i$ -th Kalman filter. If the same number of particle are used in the standard particle filter and the marginalized particle filter, the latter will theoretically provide better estimates because  $P(x_t^n | Y_t)$  lives in a smaller dimension than  $P(x_t | Y_t)$ .

The marginalized particle filter algorithm is summarized in Table 3.

## 3 MARGINALIZED PARTICLE FILTER FOR ATTITUDE ESTIMATION

Several rotation parameterizations have been studied, such as Euler angles, unit quaternions, Rodrigues parameters, modified Rodrigues parameters [15]. In this work, we use unit quaternions as they provide a globally non-singular way to represent rotations. Furthermore, we will show shortly that this parameterization will significantly simplify the MPF formulation.

### 3.1 Quaternion kinematics

The unit quaternion is used to describe the spacecraft attitude,

$$q = \begin{bmatrix} \rho \\ q_4 \end{bmatrix} = \begin{bmatrix} \hat{e} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \quad (8)$$

<b>Initialization</b>	initialize the particles (nonlinear state), $x_{0 -1}^{n,(i)} \sim P(x_0^n), \quad \forall i = 1, \dots, N$ initialize the means and covariances of the linear state $\{x_{0 -1}^{l,(i)}, P_{0 -1}^{(i)}\} = \{\bar{x}_0^l, \bar{P}_0\}$
<b>Measurement</b>	measure $y_t$
<b>PF Update</b>	assign weights $w_t^{(i)}$ to each particle, $w_t^{(i)} \propto P(y_t   x_{t t-1}^{n,(i)}, x_{t t-1}^{l,(i)}) w_{t-1}^{(i)}$ and $\sum_{i=1}^N w_t^{(i)} = 1$
<b>KF Update for <math>z_t^1</math></b>	$h_t = h_t(x_{t t-1}^{n,(i)}), C_t = C_t(x_{t t-1}^{n,(i)})$ $M_t = C_t P_{t t-1}^{(i)} C_t^T + R_t$ $K_t = P_{t t-1}^{(i)} C_t^T M_t^{-1}$ $x_{t t}^{l,(i)} = x_{t t-1}^{l,(i)} + K_t (y_t - h_t - C_t x_{t t-1}^{l,(i)})$ $P_{t t}^{(i)} = P_{t t-1}^{(i)} - K_t M_t K_t^T$
<b>Estimate</b>	$\hat{x}_{t t}^n = \sum_{i=1}^N w_t^{(i)} x_{t t-1}^{n,(i)}$ $\hat{x}_{t t}^l = \sum_{i=1}^N w_t^{(i)} x_{t t}^{l,(i)}$
<b>Resampling</b>	if $1 / \sum_{i=1}^N w_t^{(i)2} < N_{threshold}$ , resample $N$ nonlinear particles so that, $Pr(x_{t t}^{n,(i)} = x_{t t-1}^{n,(j)}) = w_t^{(j)}$ reweight $w_t^{(i)} = 1/N$
<b>PF Propagation</b>	sample $x_{t+1 t}^{n,(i)}$ , $x_{t+1 t}^{n,(i)} \sim P(x_{t+1}^n   x_{t t}^{n,(i)}, x_{t t}^{l,(i)})$
<b>KF Update for <math>z_t^2</math></b>	similar to the procedure for $z_t^1$
<b>KF Propagation</b>	$x_{t+1 t}^{l,(i)} = f_t^l + A_t^l x_{t t}^{l,(i)}$ $P_{t+1 t}^{(i)} = A_t^l P_{t t}^{(i)} A_t^{lT} + G_t^l Q_t^l G_t^{lT}$

Table 3. The marginalized particle filter algorithm, where it is assumed that  $w_t^l$  is uncorrelated with  $w_t^n$ .

where  $\hat{e}$  is a unit vector representing the axis of rotation,  $\theta$  is the angle of rotation from the reference frame to the body-fixed frame. It has to satisfy the following unity norm constraint,

$$\|q\|_2^2 = \rho^T \rho + q_4^2 = 1 \quad (9)$$

because it uses four parameters to describe three degrees of freedom. The quaternion kinematics is given to be,

$$\dot{q} = \frac{1}{2} \Xi(q) \omega$$

$$\Xi(q) = \begin{bmatrix} q_4 I_{3 \times 3} + [\rho \times] \\ -\rho^T \end{bmatrix} = \begin{bmatrix} q_4 & q_3 & -q_2 \\ -q_3 & q_4 & q_1 \\ q_2 & -q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \quad (10)$$

The matrix  $\Xi(\bullet)$  obeys the following property,

$$\Xi^T(q) \Xi(q) = I_{3 \times 3}, \quad \forall q \in \mathbb{R}^4 \quad (11)$$

The measurement of the strap-down gyroscope is usually mathematically written as,

$$\begin{aligned} \tilde{\omega} &= \omega + \beta + \zeta \\ \dot{\beta} &= \eta \\ E\left[\begin{bmatrix} \zeta \\ \eta \end{bmatrix}\right] &= 0 \\ E\left[\begin{bmatrix} \zeta \\ \eta \end{bmatrix} \begin{bmatrix} \zeta^T & \eta^T \end{bmatrix}\right] &= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = Q \end{aligned} \quad (12)$$

$\zeta, \eta$  are the angle random walk (ARW) and the rate random walk (RRW) respectively.  $\beta, \tilde{\omega}$ , and  $\omega$  are the gyro bias, the measured angular velocity, and the true angular velocity, respectively. The vector measurement model is usually written as,

$$y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix} = \begin{bmatrix} A(q)r_1 \\ A(q)r_2 \\ \vdots \\ A(q)r_M \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \quad (13)$$

$$E\left[\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix}\right] = 0, \quad E\left[\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \begin{bmatrix} v_1^T & v_2^T & \dots & v_M^T \end{bmatrix}\right] = R$$

where  $A(q)$  converts quaternions to rotation matrices,  $\{r_i, b_i\}$  is a pair of vector measurements expressed in the reference frame

and the body frame respectively,  $M$  is the number of vector measurements. If we choose the state of the particle filter to be  $x = [q^T, \beta^T]^T$ , the discretized filter dynamics can be written as,

$$\begin{aligned} q_{t+1} &= q_t + \frac{\Delta t}{2} \Xi(q_t) \tilde{\omega}_t - \frac{\Delta t}{2} \Xi(q_t) \beta_t - \frac{\Delta t}{2} \Xi(q_t) \zeta_t \\ \beta_{t+1} &= \beta_t + \Delta t \cdot \eta_t \end{aligned} \quad (14)$$

where a zero-order hold is used, and the second and higher order terms are neglected in the matrix exponential. Quaternion normalization should be performed at each step. The process and measurement noise covariances in the discrete-time model are

$$\bar{Q} = Q/\Delta t, \quad \bar{R} = R/\Delta t \quad (15)$$

### 3.2 MPF formulation

It can be readily seen that the filter dynamics in Eqn. (14), and the measurement model in Eqn. (13) are in the form of the marginalized particle filter model (Eqn. (5)). In particular, the state partition is

$$x_t = \begin{bmatrix} x_t^n \\ x_t^l \end{bmatrix} = \begin{bmatrix} q_t \\ \beta_t \end{bmatrix} \quad (16)$$

Matching with the formulation in these two equations,

$$\begin{aligned} f_t^n &= q_t + \frac{\Delta t}{2} \Xi(q_t) \tilde{\omega} \\ A_t^n &= G_t^n = -\frac{\Delta t}{2} \Xi(q_t) \\ f_t^l &= 0_{3 \times 1} \\ A_t^l &= G_t^l / \Delta t = I_{3 \times 3} \\ h_t &= \begin{bmatrix} A(q_t) r_1 \\ A(q_t) r_2 \\ \vdots \\ A(q_t) r_M \end{bmatrix} \\ C_t &= 0_{3M \times 3} \end{aligned} \quad (17)$$

A direct implementation of Table 3 with the definitions of the matrices above will lead to the MPF formulation for attitude estimation. However, there are several important features inherent in this model, which can be partially seen from the Bayes' net depicted in Figure 2. Further exploitation of the underlying linear structure leads to a significantly simplified MPF formulation. The modifications are done in the following steps: (1) KF update for  $z_t^1$ , (2) PF propagation, (3) KF update for  $z_t^2$ , (4) KF propagation. The detailed MPF procedure is presented in the following subsections:

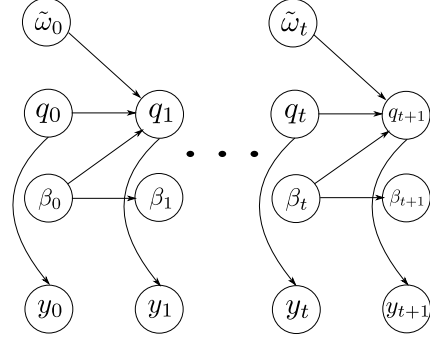


Figure 2. The Bayes' net representation of the attitude estimation problem

**3.2.1 PF Update** After the most recent vector measurement  $y_t$  is obtained, the importance weights are calculated according to,

$$\begin{aligned} e_t^{(i)} &= y_t - \begin{bmatrix} A(q_t^{(i)}) r_1 \\ A(q_t^{(i)}) r_2 \\ \vdots \\ A(q_t^{(i)}) r_M \end{bmatrix} \\ w_t^{(i)} &\propto \exp \left\{ -\frac{1}{2} e_t^{(i)T} \bar{R}^{-1} e_t^{(i)} \right\} w_{t-1}^{(i)} \end{aligned} \quad (18)$$

where  $q_t^{(i)}$  is generated from the importance sampling function which will be discussed shortly. The importance weights should be normalized after this step.

**3.2.2 Estimate** The a-posteriori bias estimate  $\hat{\beta}_t$  can be obtained by the weighted average.

$$\hat{\beta}_t = \sum_{i=1}^N w_t^{(i)} \beta_{t|t}^{(i)} \quad (19)$$

However, because of the unit norm constraint and the sign ambiguity, the weighted average estimate for quaternions is not optimal. Following Markley *et. al.* [16], the optimal average quaternion is defined as the maximizer of a constrained quadratic programming (equivalently, a weighted sum of the squared Frobenius norms of attitude matrix differences). Hence, the a-posteriori attitude estimate  $\hat{q}_t$  is,

$$\begin{aligned} \hat{q}_t &= \arg \max_q q^T L q \\ \text{subject to} \quad & q^T q = 1 \\ \text{where} \quad & L = \sum_{i=1}^N w_t^{(i)} q_t^{(i)} q_t^{(i)T} \end{aligned} \quad (20)$$

The maximization problem can be solved analytically.  $\hat{q}_t$  is an eigenvector of  $L$  corresponding to the maximum eigenvalue.

**3.2.3 KF Update for  $z_t^1$**  The measurement  $y_t$  does not contain any information about the linear state variable  $\beta_t$ . The corresponding KF update cannot be used and thus left out in the algorithm.

**3.2.4 PF propagation** The conditional a-priori distribution of the nonlinear state variables is,

$$P(q_{t+1}|q_t^{(i)}, Y_t) = \mathcal{N}\left(q_t^{(i)} + \frac{\Delta t}{2}\Xi(q_t^{(i)})(\tilde{\omega}_t - \beta_{t|t-1}^{(i)}), \frac{\Delta t^2}{4}\Xi(q_t^{(i)})(P_{t|t-1}^{(i)} + \bar{Q}_1)\Xi^T(q_t^{(i)})\right) \quad (21)$$

where  $\mathcal{N}$  represents a normal distribution.  $P(q_{t+1}|q_t^{(i)}, Y_t)$  is actually the importance sampling function in the MPF. One will instantiate  $N$  particles  $q_{t+1}^{(i)}$  from this distribution. Equivalently,  $q_{t+1}^{(i)}$  can also be generated by,

$$q_{t+1}^{(i)} = q_t^{(i)} + \frac{\Delta t}{2}\Xi(q_t^{(i)})(\tilde{\omega}_t - \beta_{t|t}^{(i)} - v) \quad (22)$$

$$v \sim \mathcal{N}(0_{3 \times 1}, P_{t|t}^{(i)} + \bar{Q}_1)$$

**3.2.5 KF Update for  $z_t^2$**  The second measurement  $z_t^2$  is important because it is the only way that information in  $y_t$  can be incorporated in the linear state variable. Based on the following measurement equation

$$z_t^2 = q_{t+1}^{(i)} - q_t^{(i)} - \frac{\Delta t}{2}\Xi(q_t^{(i)})\tilde{\omega}_t = -\frac{\Delta t}{2}\Xi(q_t^{(i)})\beta_t - \frac{\Delta t}{2}\Xi(q_t^{(i)})v \quad (23)$$

KF update is performed,

$$\beta_{t|t}^{(i)} = \beta_{t|t-1}^{(i)} + K_t(z_t^2 + \frac{\Delta t}{2}\Xi(q_t^{(i)})\beta_{t|t-1}^{(i)})$$

$$P_{t|t}^{(i)} = P_{t|t-1}^{(i)} - K_t M_t K_t^T$$

$$M_t = \frac{\Delta t^2}{4}\Xi(q_t^{(i)})(P_{t|t-1}^{(i)} + \bar{Q}_1)\Xi^T(q_t^{(i)})$$

$$K_t = -\frac{\Delta t}{2}P_{t|t-1}^{(i)}\Xi^T(q_t^{(i)})M_t^\dagger \quad (24)$$

where the pseudo-inverse of  $M_t$ , denoted as  $M_t^\dagger$  is used in calculating the KF gain. Using Eqn. (11),  $M_t^\dagger$  is found to be,

$$M_t^\dagger = \frac{4}{\Delta t^2}\Xi(q_t^{(i)})(P_{t|t-1}^{(i)} + \bar{Q}_1)^{-1}\Xi^T(q_t^{(i)}) \quad (25)$$

Also, Eqn. (23) simplifies the innovation error of KF,

$$z_t^2 + \frac{\Delta t}{2}\Xi(q_t^{(i)})\beta_{t|t}^{(i)} = -\frac{\Delta t}{2}\Xi(q_t^{(i)})v^{(i)} \quad (26)$$

where  $v^{(i)}$  is the instantiation used to generate  $q_{t+1}^{(i)}$ . With those, the KF update can be simplified to,

Mean update:

$$\beta_{t|t}^{(i)} = \beta_{t|t-1}^{(i)} + P_{t|t-1}^{(i)}(P_{t|t-1}^{(i)} + \bar{Q}_1)^{-1}v^{(i)} \quad (27)$$

Covariance update:

$$P_{t|t}^{(i)} = P_{t|t-1}^{(i)} - P_{t|t-1}^{(i)}(P_{t|t-1}^{(i)} + \bar{Q}_1)^{-1}P_{t|t-1}^{(i)} \quad (28)$$

It should be noted that the KF update does not involve  $q_t^{(i)}$ .

**3.2.6 Kalman filter propagation** The Kalman filter propagation equations are given by,

Mean propagation:

$$\beta_{t+1|t}^{(i)} = \beta_{t|t}^{(i)} \quad (29)$$

Covariance propagation:

$$P_{t+1|t}^{(i)} = P_{t|t}^{(i)} + \Delta t^2 \bar{Q}_2 \quad (30)$$

which imply that the Kalman filter equations (27) through (30) are independent of the quaternion particles. Furthermore, if each particle's bias covariances share the same initialization  $\bar{P}_0$ , then only one, instead of  $N$ , Riccati recursions is needed i.e the particle index of the covariance can be dropped, which can lead to a substantial reduction in computational complexity.

**3.2.7 Steady-state KF for linear state** Combining Eqns. (28) and (30), we obtain the following algebraic Riccati equation of the a-priori linear state covariance matrix,

$$P_{t+1|t}^{(i)} = P_{t|t-1}^{(i)} + \Delta t^2 \bar{Q}_2 - P_{t|t-1}^{(i)}(P_{t|t-1}^{(i)} + \bar{Q}_1)^{-1}P_{t|t-1}^{(i)} \quad (31)$$

Following Kalman filter theory, the steady-state solution  $P_\infty$  is guaranteed to exist and be positive definite. Therefore, in the MPF algorithm, there is no need for covariance propagation. Moreover, the mean update (Eqn. (27)) uses the steady-state KF gain  $K_\infty = P_\infty(P_\infty + \bar{Q}_1)^{-1}$ .

	PF	Steady-state MPF
<b>Initialization</b>	Initialize particles $q_0^{(i)}, \beta_0^{(i)}$	Initialize quaternion particles $q_0^{(i)}$ $\beta_0^{(i)} = \bar{\beta}_0$ Solve the Riccati equation for $P_\infty$
<b>Update</b>	Sec. 3.2.1	same as PF
<b>Estimate</b>	Sec. 3.2.2	same as PF
<b>Resampling</b>	same as Table 2	same as PF
<b>Propagation</b>	$\omega_t^{(i)} = \tilde{\omega}_t - \beta_t^{(i)} - \eta^{(i)}$ $q_{t+1}^{(i)} = q_t^{(i)} + \frac{\Delta t}{2} \Xi(q_t^{(i)}) \omega_t^{(i)}$ $\beta_{t+1}^{(i)} = \beta_{t+1}^{(i)} + \Delta t \zeta^{(i)}$ $\eta^{(i)} \sim \mathcal{N}(0_{3 \times 1}, \bar{Q}_2), \zeta^{(i)} \sim \mathcal{N}(0_{3 \times 1}, \bar{Q}_1)$	$\omega_t^{(i)} = \tilde{\omega}_t - \beta_t^{(i)} - v^{(i)}$ $q_{t+1}^{(i)} = q_t^{(i)} + \frac{\Delta t}{2} \Xi(q_t^{(i)}) \omega_t^{(i)}$ $\beta_{t+1}^{(i)} = \beta_{t+1}^{(i)} + K_\infty v^{(i)}$ $v^{(i)} \sim \mathcal{N}(0_{3 \times 1}, P_\infty + \bar{Q}_1)$

Table 4. The algorithmic comparison of PF and steady-state MPF for attitude estimation

#### 4 Comparison with PF

In this section, we discuss the differences between the steady-state MPF and the PF for attitude estimation. An algorithmic comparison is summarized in Table 4.

In the initialization step, the PF generates particles that represent the a-priori distribution of the state. However, the steady-state MPF only generates the quaternion particles, and the bias particles are set to be the a-priori mean of the bias. As discussed in Sec. 3.2.7, one needs to solve the algebraic Riccati equation for  $P_\infty$ . The update, estimate, resampling steps for the two filters are exactly the same. In the propagation step, the PF instantiates two random variables  $\eta \sim \mathcal{N}(0_{3 \times 1}, \bar{Q}_2)$  and  $\zeta \sim \mathcal{N}(0_{3 \times 1}, \bar{Q}_1)$  in the gyroscope measurement model. The MPF only instantiates one random variable  $v \sim \mathcal{N}(0_{3 \times 1}, P_\infty + \bar{Q}_1)$ . Although the two algorithms are extremely similar, the steady-state MPF is superior to the PF.

#### 5 Simulation

In this section, we demonstrate the performance of the proposed steady-state marginalized particle filter in numerical simulations. We compare the proposed steady-state marginalized particle filter with the extended Kalman filter (multiplicative quaternion formulation, Sec. 7.2 in [5]) and the standard particle filter.

Three vector measurements are generated during the simulation. Their representations in the reference frame are,

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (32)$$

The spacecraft starts at  $q_0 = [0, 1/\sqrt{2}, 0, 1/\sqrt{2}]^T$  and has a constant angular velocity  $\omega = [-.0012, 0, 0]^T rad/sec$ . The initial bias is  $\beta_0 = [0, 0, 0]^T rad/sec$ . The noise covariances matrices are given below,

$$Q_1 = (5 \times 10^{-4})^2 \cdot I_{3 \times 3}, \quad Q_2 = (2 \times 10^{-5})^2 \cdot I_{3 \times 3} \quad (33)$$

$$R = 0.03^2 \cdot I_{9 \times 9}$$

The sampling period of all the sensors is set to 1sec. We evaluate the proposed filter in the following two cases: (i) zero initial errors, (ii) large initial errors.

##### 5.1 Zero initial errors

In this case, we assume the initial state distribution as Gaussian, with no initial attitude error and bias error, i.e.  $\hat{q}_0 = q_0, \hat{\beta}_0 = \beta_0$ . The initial attitude error covariance in terms of the roll, pitch, yaw error angles is set to,

$$\Lambda_1 = 3 \times 10^{-5} \cdot I_{3 \times 3} \quad (34)$$



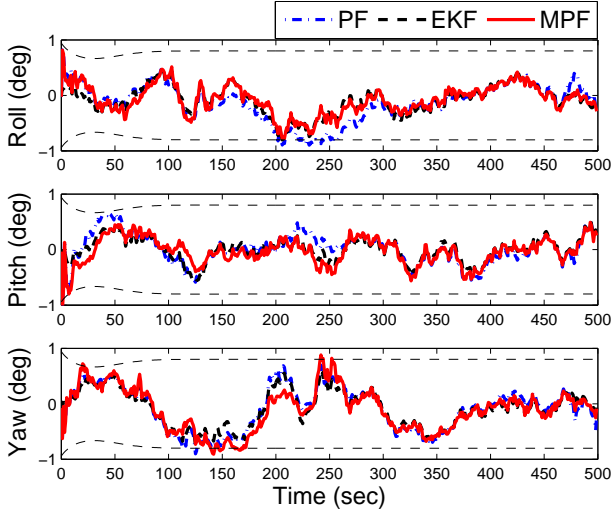


Figure 3. Case (i): plot of the roll, pitch, yaw error angles. The outer dashed lines are the  $3\text{-}\sigma$  bounds from EKF

and the initial bias covariance is,

$$\Lambda_2 = 1 \times 10^{-12} \cdot I_{3 \times 3} \quad (35)$$

Under these parameters, we compare the steady-state MPF, the PF and the EKF for a sample of 300 particles. The estimate error angles (roll, pitch, yaw) are plotted in Figure 3. The absolute error angles, calculated regardless of rotation axes, are plotted in Figure 4. From this result, we can verify that the steady-state MPF performs better than the standard PF, but not necessarily be better than the EKF. The second observation is consistent with the results reported in [9, 12]. The bias estimate errors are plotted in Figure 5, which shows comparable performance from each filter.

## 5.2 Large initial errors

This section demonstrates the global convergence properties of the proposed filter. When there is no a-priori information of the initial attitude, the EKF is infeasible. However, particle filters will still be functional by drawing samples from the uniform attitude distribution. In the simulation, the same set of initial quaternion particles is used for a fair comparison. To accommodate large uncertainties at the beginning, we follow the heuristics proposed in [9], which uses a initially large but decaying measurement standard deviation,

$$R' = (1 + 50 \exp\{-0.02t\}) \cdot 0.03^2 \cdot I_{9 \times 9} \quad (36)$$

Figure 6 shows the roll, pitch, yaw error angles. Figure 7 shows that although the absolute error angles converge 0, the two filters exhibit very different transient behaviors. Figure 8 shows that

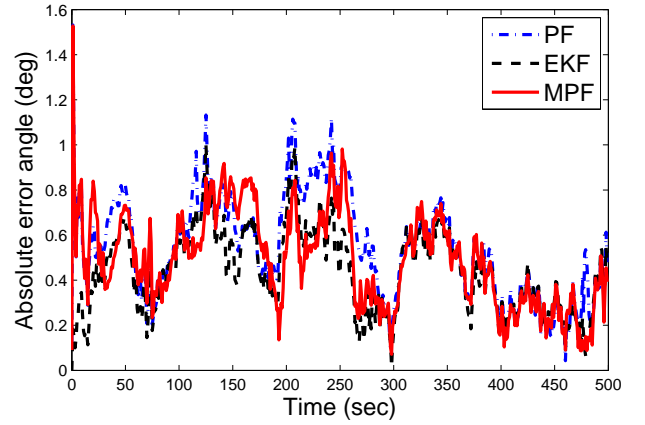


Figure 4. Case (i): plot of the absolute error angles regardless as rotation axes

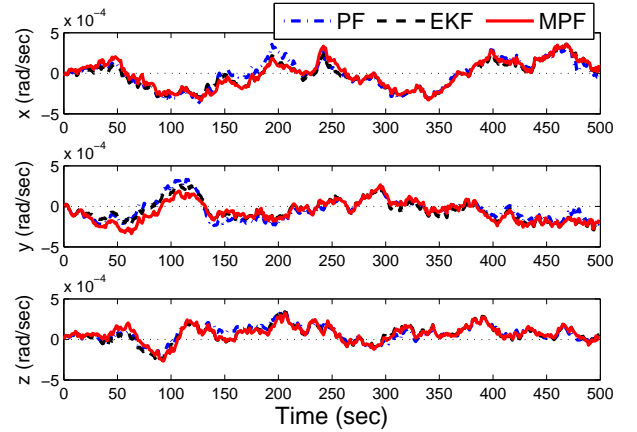


Figure 5. Case (i): plot of the bias estimate errors in three directions

the steady-state MPF has a more accurate bias estimate in this case.

## CONCLUSION

This paper derived a steady-state marginalized particle filter for sequential attitude estimation. Marginalizing out the gyroscope bias increases the estimation accuracies. By further exploiting the linear substructure, we show that the bias evolution is independent of the quaternion particles, and its covariance reaches a steady-state value, which will reduce the computation complexity. Comparison with the standard particle filter and the extended Kalman filter in numerical simulations validates the superior performance.



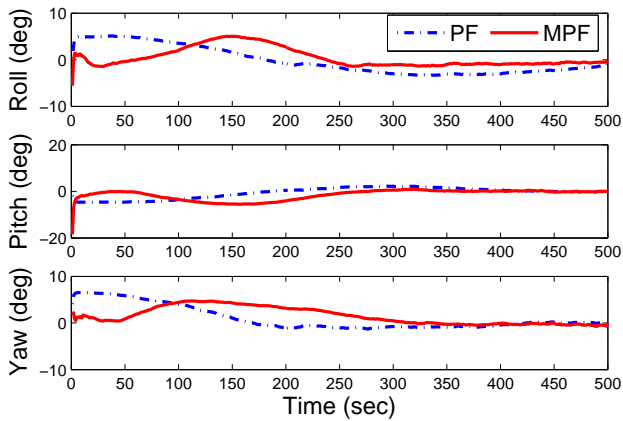


Figure 6. Case (ii): plot of the roll, pitch, yaw error angles

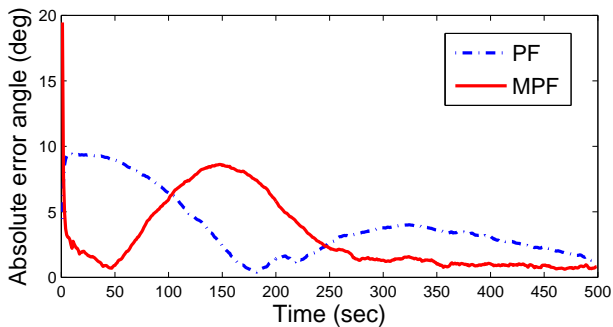


Figure 7. Case (ii): plot of the absolute error angles regardless as rotation axes

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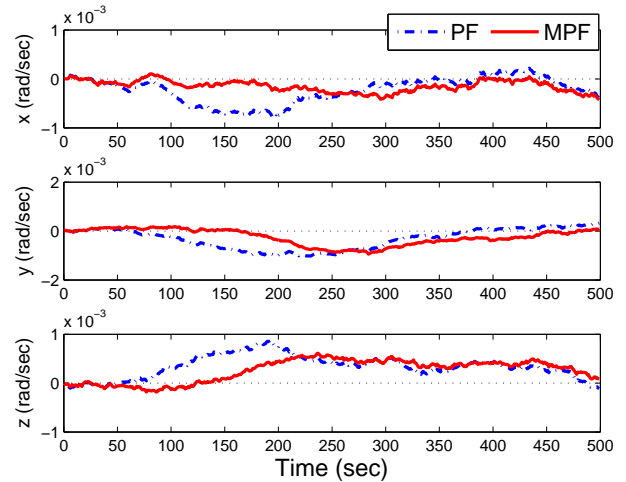


Figure 8. Case (ii): plot of the bias estimate errors in three directions

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